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The internal wave pattern produced by a sphere moving vertically in a density stratified liquid

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Experiments were conducted to test the linear theory of internal gravity waves produced in a stably stratified liquid by the steady vertical motion of a sphere. The measurements of the phase configuration in a medium whose density increased linearly with depth were made by means of a Toepler-schlieren system. The agreement between observation and prediction was found to be good.

1. Introduction

The phase configuration of internal waves produced in a stratified fluid by a body accelerating in a vertical direction from rest has been discussed by Warren (1960). The limiting form of the wave pattern existing after a sufficiently long time was sought and found to exist for an ascending body, but was not shown to exist for a descending body. Lack of knowledge of the appropriate boundary conditions far from the body prevented Warren from treating the steady problem. Lighthill (1965, 1967) has generalized the principle of Lamb (1916) by means of which the radiation condition in problems of forced oscillations can be accommodated and he has used, as one of several examples of the applications of the extended principle, the steady pattern produced by a sphere rising or falling uniformly in a stratified liquid. Although this problem does not demonstrate the full power of the extension, the wave pattern has the merit of agreeing closely with the predictions of the linear theory.

The analysis is developed, first, in terms of the Lamb–Lighthill principle and the method of stationary phase and subsequently in terms of the kinematics of wave crests, following the work of Ursell (1960). The present paper considers in detail the phase properties of the wave system and presents results of experimental measurements of the wave pattern obtained by means of a Toepler-schlieren system. The techniques used were essentially the same as those used by the authors in their investigation of the wave patterns produced by two-dimensional stationary disturbances; see Mowbray & Rarity (1967) and Mowbray (1967). It is found that the form of the crests and troughs agrees closely with the linear theory, that the patterns are fixed and identical, relative to the sphere, both when the sphere is ascending and when it is descending, and that crests do not pass through the disturbance.

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2. The dispersion relation and its implications

If ρ denotes the density, $\mathbf{q}$ the velocity, p the pressure and g the acceleration due to gravity, the equations of motion relative to a set of cylindrical polar axes, with r horizontal and y vertical and positive upwards, moving with a velocity V in the y direction, may be written in the form

$$\frac{D\rho}{Dt} = 0, \quad \nabla.(r\mathbf{q}) = 0, \quad \frac{Du}{Dt}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \quad \frac{Dv}{Dt}+\frac{1}{\rho}\frac{\partial p}{\partial y} = -g,$$

where u and v are the components of $\mathbf{q}$ conjugate to r and y respectively and D/Dt denotes

$$\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}+(v-V)\frac{\partial}{\partial y}.$$

If we denote by a suffix zero quantities with their equilibrium values and by a dash small disturbances from the equilibrium, and if we define a perturbation stream function Ψ by the relations $ru = \partial\Psi/\partial y$, $rv = -\partial\Psi/\partial r$, then the linearized equations of motion are

$$\rho_0\frac{\delta u'}{\delta t} = -\frac{\partial p'}{\partial r},$$

$$\rho_0\frac{\delta v'}{\delta t} = -\rho' g-\frac{\partial p'}{\partial y},$$

$$\frac{\delta \rho'}{\delta t}+v\frac{d\rho_0}{dy} = 0,$$

where $\delta/\delta t$ denotes $\partial/\partial t - V\,\partial/\partial y$. Eliminating the pressure gradient terms and the density, we obtain

$$\left(\frac{\delta^2}{\delta t^2}+\frac{1}{\rho_0}\frac{\delta\rho_0}{\delta t}\frac{\delta}{\delta t}\right)\left\{\frac{1}{r}\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)\right\}+\left(\frac{1}{\rho_0}\frac{\delta\, d\rho_0}{\delta t\, dy}\frac{\delta}{\delta t}+\frac{1}{\rho_0}\frac{d\rho_0}{dy}\frac{\delta^2}{\delta t^2}\right)$$
$$\times\left\{\frac{1}{r}\frac{\partial\Psi}{\partial y}\right\}-\frac{g}{\rho_0}\frac{d\rho_0}{dy}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Psi}{\partial r}\right) = 0.$$

If we look for a solution of the form

$$\Psi = \Psi^*\exp(\omega_0^2 y/2g)\exp\{i(\beta y-\omega t)\},$$

where $\omega_0^2 = -(g/\rho_0)/(d\rho_0/dy)$ denotes the square of the Väisälä–Brunt frequency, and if we consider waves whose wavelength is short compared with both the scale of density gradient $|\rho_0/(d\rho_0/dy)|$ and the scale of the gradient of $|\rho_0/(d\rho_0/dy)|$, then Ψ^* may be expressed in terms of Bessel functions of order one and argument αr where

$$S(\alpha,\beta,\omega) \equiv \omega+V\beta-\omega_0\alpha(\alpha^2+\beta^2)^{-\frac{1}{2}} = 0.$$

Thus for sufficiently large r, we may write the solution as

$$\Psi = \iint f(y,r,\alpha,\beta)\exp\{i(\alpha r+\beta y)\}\,d\alpha\,d\beta.$$

Lighthill (1965) shows how the integral is to be interpreted to satisfy the radiation condition; in the present case it reduces to the condition of Lamb (1916), that the component, in the direction of the phase velocity, of the velocity of the disturbance should be equal to the phase velocity, which is equivalent to $S(\alpha, \beta, 0) = 0$. Disturbances are propagated only in the direction of positive group velocity, that is in the directions of normals to the curve $\omega = 0$ in the senses indicated in figure 1. Thus we see that for a sphere travelling with velocity V upwards the waves are everywhere below the body.

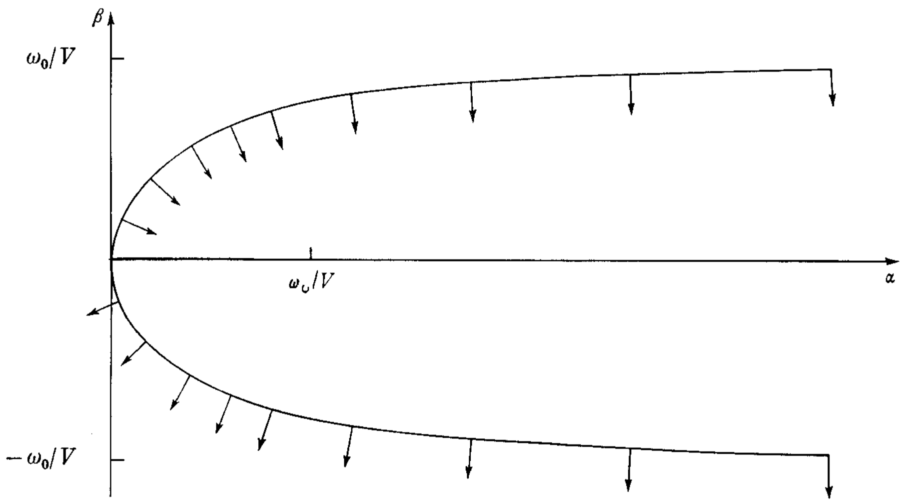

FIGURE 1. The curve $\omega = 0$; the arrows indicate the normals in the direction of $\omega > 0$.

Let us consider the phase configuration. The disturbance may now be represented by

$$\Psi = \int f(y, r, \alpha(\beta), \beta) \exp\{i(\alpha(\beta) r + \beta y)\} d\beta,$$

where $\alpha = \alpha(\beta)$ is determined from the relation $S(\alpha, \beta, 0) = 0$. We define the angle ψ to be the angle between the wave-number vector (α, β) and the vertical; hence $\sin\psi = \alpha(\alpha^2+\beta^2)^{-\frac{1}{2}}$. As in the case of two-dimensional stationary disturbances, ψ can be shown to be the angle between the tangent to the crest and the horizontal. The principle of stationary phase states that the major contribution to the integral is derived from points β at which

$$d/d\beta\{\alpha(\beta) r/y + \beta\} = 0$$

for large y, say, with $r/y = O(1)$. Curves of constant phase are given by:

$$\Phi = \alpha(\beta) r + \beta y = \text{constant},$$

or

$$\left.\begin{aligned} \omega_0 R/V\Phi &= \sqrt{\{\sigma^{-4}+\sigma^{-2}-1\}}, \\ \cot\theta &= \sigma(2-\sigma^2)(1-\sigma^2)^{-\frac{3}{2}}, \end{aligned}\right\} \tag{1}$$

where $r = R\sin\theta$, $y = -R\cos\theta$ and $\sigma = V\beta/\omega_0$ is the parameter. We note that $|\sigma| \leqslant 1$ and that as $\sigma \to 0$, $R \to \infty$ and $\theta \to \frac{1}{2}\pi$; as $|\sigma| \to 1$, $\theta \to 0$ and $R \to V\Phi/\omega_0$, which is non-zero in general. The curve has a cusp at the point $\theta = 0$, $R = V\Phi/\omega_0$; hence crests do not pass through the disturbance. Equations (1) show that, for given θ, R increases linearly with V, so that the pattern becomes more 'swept-back' with increasing V.

An equivalent result may be derived from the following, which is essentially the set of rules given by Ursell (1960) for the construction of curves of constant

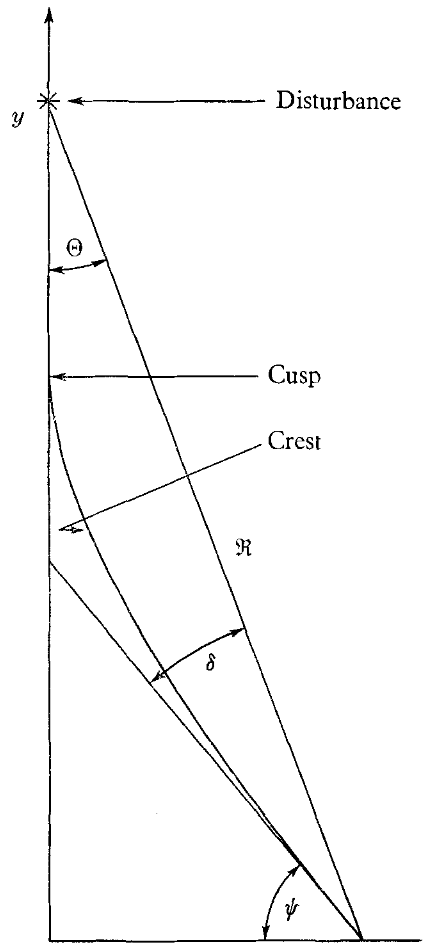

FIGURE 2. The angles Θ, ψ, δ and the radius $\mathfrak{R}$.

phase. The phase velocity $c_{\mathrm{ph}} = (\omega_0 \sin\psi)/k$ is a known function of ψ and k where $k = (\alpha^2+\beta^2)^{\frac{1}{2}}$. We must require that the pattern be steady with respect to the disturbance so that $\cos\psi = c_{\mathrm{ph}}/V$, and we may define the angle $\delta(\psi, k)$ by the relation $\tan\delta = k(d\psi/dk)$. The radius vector $\mathfrak{R}(k)$ and the angle $\Theta(k)$ are defined by

$$\mathfrak{R}(k) = A/k\sin\delta(k), \quad \Theta(k) = \tfrac{1}{2}\pi - \psi - \delta,$$

where A is a constant which changes by 2π from crest to crest. These relations define curves of constant phase. The angles are shown in figure 2; δ is the angle between the radius and the tangent and ψ is the angle between the tangent and the horizontal, as previously. It is easily shown that

$$\left.\begin{aligned} \omega_0 \Re / VA &= \sqrt{\{s^{-2} + (1+s^{-2})^2\}}, \\ \cot \Theta &= s(2+s^2), \end{aligned}\right\} \qquad (2)$$

where $s = Vk/\omega_0$ is a parameter. We note that when $\Theta \to 0$, $s \to \infty$ and $\Re \to A\,V/\omega_0$ which is our previous result with A replaced by Φ. The reader will note that since $s = Vk/\omega_0$, we may write $s = \sigma(1-\sigma^2)^{\frac{1}{2}}$. Substitution in equations (2) shows that $\Theta = \theta$ and that $\Re = R$, when $A = \Phi$, so that the results are equivalent.

This demonstrates that the two approaches, the first based essentially on group velocity, the second on the kinematics of wave crests, are equivalent. Ursell (1960) showed that the latter implied the former for systems in which the magnitude of the phase velocity is a function k only. Here the phase velocity has an explicit dependence on ψ also. The reader may verify that the result continues to hold if the group velocity is defined by the relation

$$\mathbf{c}_{\mathrm{gr}} = \mathbf{c}_{\mathrm{ph}} + k\frac{\partial}{\partial k}\mathbf{c}_{\mathrm{ph}} + (\cos\psi,\ -\sin\psi)\frac{\partial}{\partial\psi}|\mathbf{c}_{\mathrm{ph}}|,$$

where $\mathbf{k} = k(\sin\psi,\ \cos\psi)$.

3. Comparison of theory and experiment

A glass walled rectangular tank, 50 cm square by 100 cm deep, was carefully filled with salt water of varying salinity, producing a medium of linearly varying density after a sufficiently long period of time. The disturbances were produced by spheres of various sizes moved by means of thin nylon line attached to a constant speed motor, and were observed by means of a Toepler-schlieren system, similar to that employed in the experiments described by Mowbray & Rarity (1967); the reader should consult Mowbray (1967) for a detailed discussion of the techniques. The optical arrangements were such that a photograph of the undisturbed medium produced a plate of uniform contrast; lines of equal contrast on a photograph of the disturbed medium are the loci of points of constant phase in the wave system. A typical wave pattern produced by a sphere moving upwards is shown in plate 1 (*a*), for which the parameter $V/\omega_0 d = 0{\cdot}200$; d is the diameter of the sphere. Observation of the wave pattern and of ciné film of the wave pattern establishes that the crests and troughs are stationary with respect to the sphere, and confirms that waves from a sphere moving with uniform velocity appear only behind the sphere. It is also clear from plate 1 (*a*) that crests do not pass through the disturbance but appear to have a cusp at some distance behind. Figure 3 shows adjacent curves of constant phase corresponding to phases 8π increasing by multiples of 2π to 16π, where $R = 1$ has been taken, arbitrarily, as the cusp on the curve of phase 8π.

Plate 1 (*b*), shows the wave pattern produced by the same sphere as that of plate 1 (*a*), but the sphere is moving downwards and with a greater velocity, corresponding to $V/\omega_0 d = -0{\cdot}295$. Measurements of plates 1 (*a*) and (*b*) agree well with the theoretical predictions. The experiment was repeated with spheres of various diameters moving with different velocities, both negative and positive; the agreement with prediction was uniformly good throughout the range of experiments. Figure 4 shows $\omega_0 R/V\Phi$ as a function of θ together with measurements obtained from experiment; the average error in $\omega_0 R/V\Phi$ is expected to be about $\pm 0{\cdot}05$ for the curves of smallest phase and about $\pm 0{\cdot}02$ for the curve of

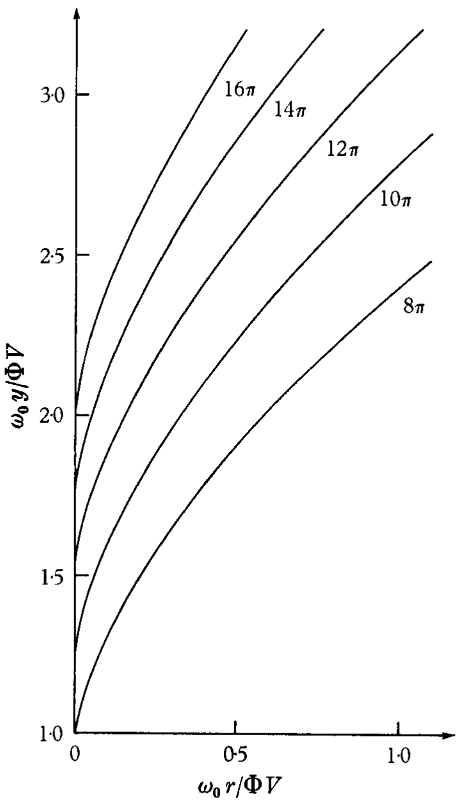

FIGURE 3. Curves of constant phase; the phases of adjacent curves differ by 2π.

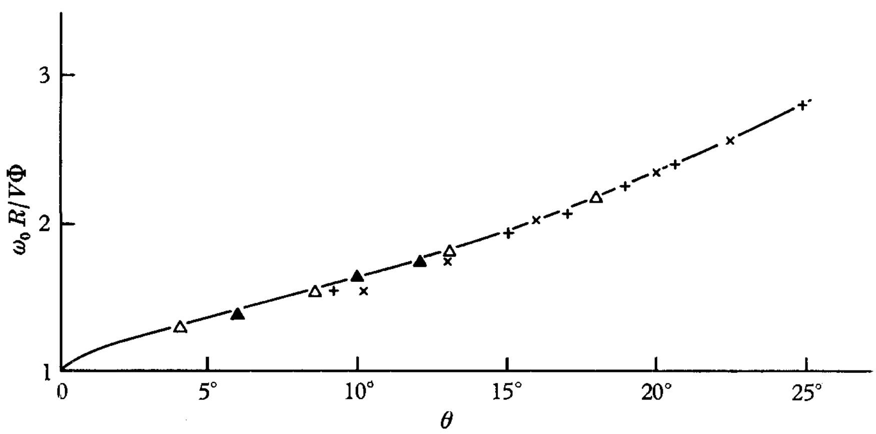

FIGURE 4. The function $\omega_0 R/V\Phi$ as a function of θ (degrees). $\times$, $\Phi/2\pi = \frac{2}{3}$; $+$, $\Phi/2\pi = 1\frac{2}{3}$; $\triangle$, $\Phi/2\pi = 2\frac{2}{3}$; $\blacktriangle$, $\Phi/2\pi = 3\frac{2}{3}$.

highest phase; this error arises by virtue of the difficulty of accurately determining points of corresponding phase on photographs of the wave pattern. The error in θ is nominally zero.

It will be observed that the wave patterns of plates 1 (*a*) and (*b*) bear some similarity to the Kelvin shipwave pattern. It is readily shown, however, that there are no double points of stationary phase and hence no envelope of waves corresponding to the wedge of semi-angle $\sin^{-1}(1/\sqrt{3})$; there is no transverse wave system. What evidence there seems to be in plates 1 (*a*) and (*b*) for the existence of an envelope is based on the form of the crest of the most recently created wave, a region in which the analysis cannot be expected to apply.

Plate 1 (*c*) shows the disturbance pattern produced by a sphere for which $V/\omega_0 d = 1{\cdot}00$. We see that there is no distinct and well-defined pattern as in plates 1 (*a*) and (*b*). The critical value of $|V/\omega_0 d|$ at which the pattern ceased to be well defined was found to be 0·9. In all experiments the Reynolds numbers of the sphere were kept below the value 200; as far as could be observed, the sphere did not shed vortices, although the non-uniform density appeared to influence markedly the behaviour of the fluid in the boundary layer. No systematic study was made of the behaviour of fluid in the boundary layer, or in the wake, other than to ensure that 'periodic vortices' were not being shed.

We mention, in passing, that reflexions of the ordinary kind were observed at the rigid side surfaces (see Mowbray & Rarity 1967); reflexions of the singular kind were also observed. The wave system to be expected from a more or less impulsive start was never observed, presumably being swamped by the steady pattern, neither were the effects of finite body size detected.

4. Conclusion

It is found that the linear theory of the phase configuration of small amplitude waves produced by a point disturbance moving uniformly in a vertical direction agrees well with the pattern observed to be produced by a moving sphere. It is confirmed that the pattern is stationary with respect to the sphere and is the same, relative to the sphere, whether the sphere is ascending or descending.

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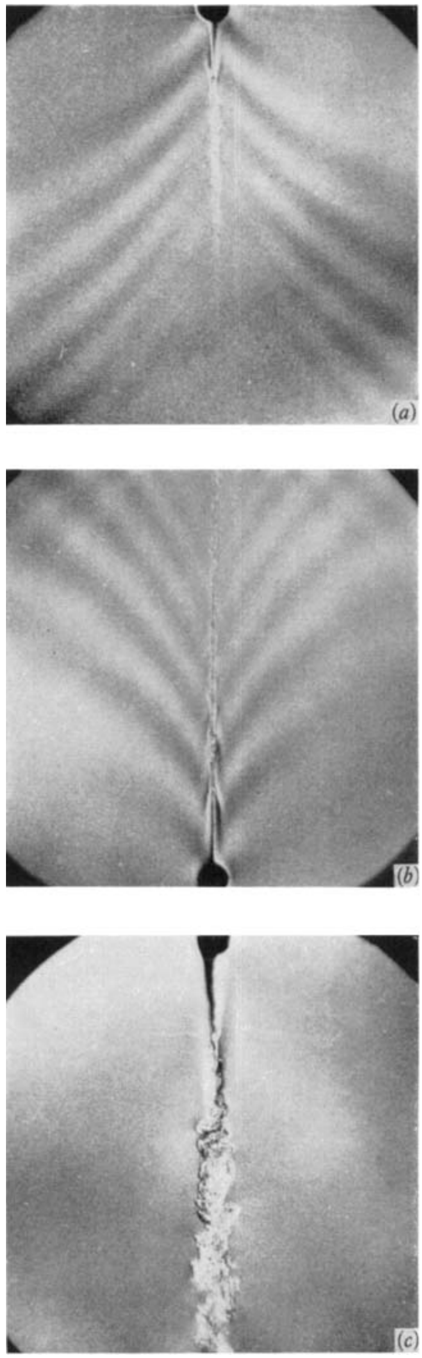

PLATE 1. (*a*) $V/\omega_0 d = 0{\cdot}200$. (*b*) $V/\omega_0 d = -0{\cdot}295$; the diameter of the sphere is the same in photographs (*a*) and (*b*). (*c*) $V/\omega_0 d = 1{\cdot}00$.